Thermal noise suppression: how much does it cost?

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Abstract

In order to stabilize the behavior of noisy systems, confining it around a
desirable state, an effort is required to suppress the intrinsic noise. This
noise suppression task entails a cost. For the important case of thermal noise
in an overdamped system, we show that the minimum cost is achieved when
the system control parameters are held constant: any additional deterministic
or random modulation produces an increase of the cost. We discuss the
implications of this phenomenon for those overdamped systems whose control
parameters are intrinsically noisy, presenting a case study based on the example
of a Brownian particle optically trapped in an oscillating potential.

PACS numbers: 05.40.−a, 02.50.−r

1. Introduction

Noisy systems are ubiquitous in natural and engineered phenomena. The presence of noise
becomes particularly evident when we move down into molecular-scale phenomena: the
thermal noise, responsible for the Brownian diffusion of particles, is omnipresent. However,
noise is also intrinsic to many macroscopic systems [1–5]: stock markets, population dynamics,
ecosystems and traffic flows, all present some degree of noise.

Even though in recent years the constructive role of noise has been appreciated in many
physical, chemical and biological phenomena—examples include stochastic resonance [6–11],
noise-induced transitions [12], noise-induced transport [13–15], stochastic resonant damping
[16]—there are many situations in which the intrinsic noise of a system is still a nuisance that
one wants to keep under control, and minimize if possible [17–19].

Noise suppression is a crucial task at all scales. Microscopic and nanoscopic phenomena
have to deal with thermal noise. Complex pricing systems, such as the Black–Scholes option

pricing model [20], have been developed for dealing with the noise present in the stock markets. Given the insolubility of the multi-body problem, noise has to be dealt with in the planning of satellites' trajectories. In all these cases, one needs to exert some kind of control on the system in order to minimize its intrinsic noise. Often these actions are controlled by some *input parameters*, which may also vary over time either deterministically or randomly.

Here we will focus on an Ornestein–Uhlenbeck equation, which describes an overdamped system. Such an equation describes a very wide class of systems, and it has successfully been applied to systems as diverse as macromolecules that follow the Hooke's law, Brownian particles, electronic devices and mesoscopic chemical reactions [21]. As a simple example let us consider the diffusion of a Brownian particle. The particle position variance is reduced if the particle is confined in a potential well. This potential well can be produced by various means: by a molecule that binds the particle, by hydrodynamic focusing or by optical or magnetic tweezers. All these means have in common that they exert a restoring force on the particle whenever it is displaced from the desired position. A tighter confinement of the particle is achieved by increasing the stiffness of the link, but a higher stiffness implicates undergoing a higher cost to run the system.

Here we analyze the cost of noise suppression in non-equilibrium systems and introduce a *cost function* to quantify the effort made to control such a system. We show that the minimum cost is achieved when the system control parameters are held constant. We find that any additional deterministic or random modulation of the control parameters entails an increase of the cost function.

2. Model

We consider a dynamical system driven by a Gaussian white random process dB_t (Wiener process), whose state s_t freely evolves according to $ds_t = dB_t$. By introducing a restoring force, characterized by a constant stiffness $\bar{k}$, the system can be forced to fluctuate around a state $\bar{a}$, with variance $\sigma_s^2 = 1/2\bar{k}$. The stochastic dynamics of the system (known as the Ornstein–Uhlenbeck process) is described by

$$ds_t = -\bar{k}(s_t - \bar{a})\,dt + dB_t. \tag{1}$$

We can now analyze the effect of fluctuating parameters by letting the mean state a_t and the stiffness k_t—and therefore the intrinsic fluctuations $\sigma_s^2 = 1/2k_t$—as generic processes independent of dB_t and possibly dependent between themselves with $E[a_t] = 0$ and $E[k_t] > 0$, where $E[\cdot]$ denotes the expected value. These two conditions guarantee the long-term stability of the system. In the case of a Brownian particle, they assure that the particle will not eventually escape from the potential. The former condition ($E[a_t] = 0$), in particular, signifies that the potential keeps on oscillating around the state $s = 0$; the latter ($E[k_t] > 0$) that the average stiffness is positive. In particular, all our conclusions apply to the case in which a_t and k_t are functions of an Ornstein–Uhlenbeck process. Note also that the conclusions also apply to the case in which a_t and k_t are deterministic functions, considering that these are a special case of the random function. The system time evolution obeys the equation

$$ds_t = -k_t(s_t - a_t)\,dt + dB_t. \tag{2}$$

As we have already mentioned, the diffusion of a Brownian particle in a time-varying potential is an example of processes described by (2) [16, 22, 23]. A free particle diffuses in such a way that the variance of its position grows linearly with time. The diffusion process can be partially suppressed by confining the particle in a potential well which, for example, can be produced by an optical trap [24]. In the presence of an optical trap, whose center and

stiffness oscillate, the particle position obeys equation (2), with a_t being the center of the trap and k_t its stiffness.

With our analysis we aim at finding the output variance of the state of the system described by (2) when the parameters vary over time in an arbitrary fashion. In particular we will identify four contributions in the total variance: intrinsic variance $\sigma_{s(i)}^2$, stiffness variance $\sigma_{s(k)}^2$, equilibrium variance $\sigma_{s(a)}^2$, interplay covariance $\rho_{s(ak)}$. All these cases may be studied experimentally.

Instead of attacking the general case directly, we will proceed by steps, starting by investigating some limiting cases. This will permit us to build up the necessary intuition and to gain useful insights into the phenomenon.

We note that in the cases we study the Itô and Stratonovich approaches to stochastic integration are mathematically equivalent, because the diffusion term is constant [25, pp 35–37]. Here we are considering the system's steady state, but the conclusions apply with little variations also to the transient.

2.1. The stationary case ($a_t \equiv 0$ and $k_t \equiv \bar{k}$)

The simplest case is when the equilibrium position of the harmonic potential does not oscillate ($a_t \equiv 0$) and its stiffness is kept constant ($k_t \equiv \bar{k} > 0$). This is the benchmark against which all other results will be compared. Equation (2) simplifies as

$$ds_t = -\bar{k}s_t\,dt + dB_t. \tag{3}$$

Its solution can be found by multiplying by the integrating factor $e^{\bar{k}t}$ and comparing with $d(e^{\bar{k}t}s_t) = \bar{k}\,e^{\bar{k}t}s_t\,dt + e^{\bar{k}t}\,ds_t$. The solution is

$$s_t = \underbrace{e^{-\bar{k}t}x_0}_{\to 0} + e^{-\bar{k}t}\int_0^t e^{\bar{k}u}\,dB_u \to \int_0^t e^{-\bar{k}(t-u)}\,dB_u, \tag{4}$$

where the limit has been taken for large t.

It follows that the mean of the system is $E[s_t] = 0$ because it is an Itô integral, and its variance is

$$E[s_t^2] = \int_0^t E[e^{-2\bar{k}(t-u)}]\,du = \int_0^t e^{-2\bar{k}(t-u)}\,du = \frac{1 - e^{-2\bar{k}t}}{2\bar{k}} \to \frac{1}{2\bar{k}}, \tag{5}$$

where the Itô isometry $E\big[\big(\int_0^t f(u,\omega)\,dB_u\big)^2\big] = E\big[\int_0^t f^2(u,\omega)\,du\big]$ has been used [25, 26].

We can therefore identify the *intrinsic variance* as a contribution to the total variance of the system

$$\sigma_{s(i)}^2 = \frac{1}{2\bar{k}}. \tag{6}$$

As we will see this is the minimum variance that can be achieved for a given value of the cost function, i.e. for a given $\bar{k}$.

2.2. Fluctuating k_t ($a_t \equiv 0$)

When $a_t \equiv 0$, equation (2) simplifies as

$$ds_t = -k_t s_t\,dt + dB_t, \tag{7}$$

where k_t is an Itô process independent of B_t. Again the solution can be calculated by multiplying by the integrating factor $e^{\int_0^t k_u \, du}$ and comparing with $d\big(e^{\int_0^t k_u \, du} s_t\big) = k_t \, e^{\int_0^t k_u \, du} s_t \, dt + e^{\int_0^t k_u \, du} \, ds_t$. Its solution is

$$s_t = \underbrace{e^{-\int_0^t k_u \, du} x_0}_{\to 0} + e^{-\int_0^t k_u \, du} \int_0^t e^{\int_0^u k_v \, dv} \, dB_u \to \int_0^t e^{-\int_u^t k_v \, dv} \, dB_u, \tag{8}$$

where the first term vanishes for large t because $E[k_t] = \bar{k} > 0$.

We can therefore calculate the mean and the variance of the system.

$$E[s_t] = E\left[\int_0^t e^{-\int_u^t k_v \, dv} \, dB_u\right] = 0, \tag{9}$$

because it is an Itô integral, and

$$E\big[s_t^2\big] = \int_0^t E\big[e^{-2\int_u^t k_v \, dv}\big] \, du \geqslant \int_0^t e^{-2\bar{k}(t-u)} \, du = \sigma_{s(i)}^2 \tag{10}$$

where we have used the Itô isometry and Jensen inequality [27] $E\big[e^{-2\int_u^t k_v \, dv} \, du\big] \geqslant e^{-2E[k_t](t-u)} \, du$ integrated over time with $E[k_t] = \bar{k}$.

We can now identify *the stiffness variance* as a contribution to the total system variance, caused by the variation of the stiffness

$$\sigma_{s(k)}^2 = \int_0^t e^{-2\bar{k}(t-u)} E\big[e^{-2\int_u^t (k_v - \bar{k}) \, dv} - 1\big] \, du. \tag{11}$$

2.3. Fluctuating a_t ($k_t \equiv \bar{k}$)

The case when the equilibrium position of the potential a_t is oscillating, while $k_t \equiv \bar{k}$ remains constant, was investigated both theoretically and experimentally in [16]. However, a different approach was applied there and it can be useful to obtain the same result expressed in the current formalism. Equation (2) becomes

$$ds_t = -\bar{k}(s_t - a_t) \, dt + dB_t. \tag{12}$$

It can again be solved by multiplying by the integrating factor $e^{\bar{k}t}$ and comparing with $d(e^{\bar{k}t} s_t) = \bar{k} \, e^{\bar{k}t} s_t \, dt + e^{\bar{k}t} \, ds_t$. Its solution is

$$s_t = \underbrace{e^{-\bar{k}t} s_0}_{\to 0} + \bar{k} \, e^{-\bar{k}t} \int_0^t e^{\bar{k}u} a_u \, du + e^{-\bar{k}t} \int_0^t e^{\bar{k}u} \, dB_u \tag{13}$$

$$= \bar{k} \int_0^t e^{\bar{k}(t-u)} a_u \, du + \int_0^t e^{\bar{k}(t-u)} \, dB_u. \tag{14}$$

Since the process a_t is independent of B_s, in the calculation of the variance of s_t the contributions of the two integrals can be separated,

$$E\big[s_t^2\big] = \sigma_{s(i)}^2 + \sigma_{s(a)}^2, \tag{15}$$

where the equilibrium variance

$$\sigma_{s(a)}^2 = \bar{k}^2 E\left[\left(\int_0^t e^{\bar{k}(t-u)} a_u \, du\right)^2\right] \tag{16}$$

is the contribution to the variance of the system due to the oscillation of the equilibrium position of the potential. The second term is the one corresponding to the stationary state. More details and a discussion of how this effect produces the stochastic resonant damping can be found in [16].

2.4. General case—fluctuating a_t and k_t

In the general case given by equation (2), again the solution can be calculated by multiplying by the integrating factor $e^{\int_0^t k_u \, du}$ and comparing with $d\left(e^{\int_0^t k_u \, du} s_t\right) = k_t \, e^{\int_0^t k_u \, du} s_t \, dt + e^{\int_0^t k_u \, du} \, ds_t$. The general solution is

$$s_t = \underbrace{e^{-\int_0^t k_u \, du} s_0}_{\to 0} + e^{-\int_0^t k_u \, du} \int_0^t e^{\int_0^u k_v \, dv} k_u a_u \, du + e^{-\int_0^t k_u \, du} \int_0^t e^{\int_0^u k_v \, dv} \, dB_u \tag{17}$$

$$= \int_0^t e^{\int_u^t k_v \, dv} k_u a_u \, du + \int_0^t e^{\int_u^t k_v \, dv} \, dB_u. \tag{18}$$

For large t following a procedure similar to the previous cases the variance of the system in the general case is given by

$$E\left[s_t^2\right] = E\left[\left(\int_0^t e^{-\int_u^t k_v \, dv} \, dB_u\right)^2\right] + E\left[\left(\int_0^t e^{\int_u^t k_v \, dv} k_u a_u \, du\right)^2\right] \tag{19}$$

$$= \sigma_{s(i)}^2 + \sigma_{s(k)}^2 + E\left[\left(\int_0^t e^{\int_u^t k_v \, dv} k_u a_u \, du\right)^2\right] \tag{20}$$

$$= \sigma_{s(i)}^2 + \sigma_{s(k)}^2 + \rho_{s(ak)} + \sigma_{s(a)}^2, \tag{21}$$

where

$$\rho_{s(ak)} = E\left[\left(\int_0^t e^{\int_u^t k_v \, dv} k_u a_u \, du\right)^2\right] - \bar{k}^2 E\left[\left(\int_0^t e^{\bar{k}(t-u)} a_u \, du\right)^2\right] \tag{22}$$

is the *interplay covariance*, which can be either positive or negative. However, the total variance is always larger than the intrinsic variance, since, as can be seen from equation (21), the overall contribution due to the oscillation of the stiffness and the equilibrium position is always positive

$$\sigma_{s(k)}^2 + \rho_{s(ak)} + \sigma_{s(a)}^2 > 0. \tag{23}$$

2.5. Cost function

As equation (19) shows that one can use different protocols in order to change the output variance of a given intrinsically noisy system by means of the modulation of the system parameters. Now we do the next step and ask the key question of this study: how one can compare the protocols from the point of view an effort applied to change the output variance? To deal with such a question mathematically, we suggest introducing a *cost function*.

The idea of a cost function, sometimes referred to as an *objective function*, is very well established in the fields of economic optimization [28] and in engineering [29]: it permits one to compare the performance of systems that work under different conditions. Typically for a given cost one looks for the parameters that provide the best performance (the smallest variance in our case). We introduce here the idea and the importance of a cost function in the study of the confinement of overdamped systems. An appropriate cost function needs to describe the overall effort made in a system to achieve its confinement.

For a stationary system, the stiffness $\bar{k}$ fully describes the confinement effort. Indeed, as we have seen, the output variance of a stationary system $\sigma_{s(i)}^2$ is inversely proportional to the stiffness. Therefore to define the cost function as $\mathcal{C} = \bar{k}$ seems rather natural.

We introduce a similar cost function for systems whose parameters are modulated over time. First, as was shown in [16], the modulation of the mean state a_t does not affect the effort made to confine the system; we therefore need to consider only the modulation of the stiffness in order to introduce the cost function. For the systems where the stiffness k_t varies over time, we suggest using as a cost function the average value of the stiffness

$$\bar{\mathcal{C}} = E[k_t]. \tag{24}$$

As seen the cost function of a stationary system calculated using this formula has the same value as it was defined before.

Let us compare the variance of a stationary system and the same system but with modulated parameters, assuming that the cost functions are equal for both systems. To maintain the cost function of the system constant, we must keep the average stiffness invariant $E[k_t] = \bar{k}$. From this condition it follows that the intrinsic variance $\sigma_{s(i)}^2$ is also constant, and, as a straightforward consequence of equation (21), it coincides with the minimum variance. This means that for a given value of the cost function the output variance of the system with modulated parameters is bounded by its intrinsic variance

$$E\left[s_t^2\right] = \sigma_{s(i)}^2 + \sigma_{s(k)}^2 + \rho_{s(ak)} + \sigma_{s(a)}^2 > \sigma_{s(i)}^2. \tag{25}$$

This can equivalently be stated as the fact that, for a given cost function, any additional deterministic or random modulation implies a larger system variance. For a given value of the cost function the minimum of the variance is achieved when the control parameters are constant. From another point of view this means also that for a given system variance, any additional deterministic or random modulation produces an increase of the cost function.

2.6. Notes on nonlinear potentials

The discussion so far has been centered on a fluctuating linear potential $V_t^{\text{lin}}(s) = \frac{k_t^2}{2}(s - a_t)^2$, which leads to the well-established Ornstein–Ulhenbeck equation. This is a good model to describe a wide range of phenomena from many fields of physics, biology and economy. In particular, the Ornstein–Ulhenbeck model is a first-order-approximated description of the behavior of an intrinsically noisy system and it is the reference model for a Brownian particle kept in a trapping potential.

However, there are effects, such as stochastic resonance, that require nonlinear potentials to manifest themselves. Therefore it is worthy to briefly extend the discussion presented here to the case of systems characterized by a nonlinear potential. We can consider such potential as a series expansion

$$V_t^{nl}(s) = \frac{k_t^2}{2} \sum_{n=1}^{+\infty} [\alpha_n(s - a_t)^{2n} + \beta_n(s - a_t)^{1/(2n)}]. \tag{26}$$

Note that we consider a potential around an equilibrium position. In this paper we have studied in detail the linear case, i.e. $\alpha_n = \delta_{1n}$ and $\beta_n = 0$.

In the case nonlinear terms are relevant, we might still define the cost as a function of k_t, as defined in equation (24). However, the general conclusion of this paper may not apply, i.e. there are nonlinear potentials for which the presence of a fluctuation can produce a decrease of the variance. This is true, for example, for a potential with only $\beta_2 > 0$ in the limit of fast fluctuations. The exact behavior of this is an interesting argument that is being investigated and will be subject of a future work.

3. Brownian particle example

Experimental results related to our study were presented in [22] where the dynamics of a Brownian particle held in an optical trap with modulated position and stiffness was measured. For a given experimental configuration the stiffness of the (stationary) trap, and, therefore, the achieved confinement is proportional to the optical power P used to create the trap ($\bar{k} \propto P$). Therefore, a higher confinement requires an higher optical power and the cost function of the system is defined as $\mathcal{C} = \bar{k} \propto P$.

In the presence of an optical trap, whose center and stiffness oscillate, the particle position obeys equation (2), with a_t being the center of the trap and $k_t = 1/\sigma_s^2$ its stiffness. In this case, the cost function is $\tilde{\mathcal{C}} = E[k_t] \propto E[P(k_t)]$, where the optical power $P(k_t)$ that must be used to create the optical trap also fluctuates.

By using a specific protocol of modulation of the trap parameters a reduced variance of the observed particle position as compared to a stationary trap was observed. We analyzed the experimental data by calculating the cost function for the stationary and modulated traps. When a stationary trap was used (stiffness $k^{(0)} = 3.7 \, \mathrm{pN} \, \mu\mathrm{m}^{-1}$) the output variance of the particle position was $\sigma_{\mathrm{out}}^2 = 1087 \, \mathrm{nm}^2$ (figure 1(b) of [22]). With the oscillating trap, the output position variance was indeed reduced to $\sigma_{\mathrm{out}}^2 = 764 \, \mathrm{nm}^2$ (figure 1(c) of [22]). From the data presented in [22] we calculated the average stiffness as $\bar{k} = E[k_t] = \int_{-\infty}^{+\infty} k p_k(k) \, \mathrm{d}k$, therefore substituting the time average with the average over $p_k(k)$ the probability density of k_t (figure 1(b) of [22]), which can be computed from equations (2) and (3) of [22]. The average stiffness of the modulated trap (and therefore the average optical power introduced in the system and the cost function of the process of the noise reduction) was found considerably bigger than that in the stationary case ($\bar{k} = 6.8 \, \mathrm{pN} \, \mu\mathrm{m}^{-1}$). Therefore, we can conclude that the higher confinement of the particle position in the trap was achieved not due to the addition of noise to the trapping parameters, but also due to the higher average trapping power, while the added noise slightly increases the output variance. We note that a stationary trap with such average power and the same experimental configuration would produce an even smaller output variance than the modulated trap did.

4. Conclusions

We have shown that to suppress the intrinsic noise of a system entails a cost. For a noisy system in the overdamped regime controlled by a fluctuating input parameter, we have found out that the minimum cost is achieved when the system control parameters are held constant: any additional deterministic or random modulation of the parameters yields an increase of the cost function. It is not possible to reduce the output noise of a system below a threshold value, corresponding to constant input parameters without increasing the cost function of the system. This has important implications both from the fundamental point of view, in order to understand many natural phenomena—for example, the natural optimization of cellular molecular phenomena—and from the engineering one, where it can give a guidance in the management of the intrinsic noise of a system.

Acknowledgments

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